حل تمرین های هفته چهارم درس نسبیت

الف) برای اثبات فضای برداری بودن $\mathbb{R}^n_a$ باید موارد زیر را بررسی کنیم:

Associativity of addition: $v_a + (u_a + w_a) = v_a + (u+w)_a = ((v+(u+w)))_a = ((v+u)+w)_a$

$$= (v+u)_a + w_a = (v_a + u_a) + w_a \quad \checkmark$$

Commutativity of addition: $u_a + v_a = (u+v)_a = (v+u)_a = v_a + u_a \quad \checkmark$

Identity element of addition: $0_a + v_a = (0+v)_a = v_a \quad \checkmark$

Inverse elements of addition: $v_a + (-v)_a = (v+(-v))_a = 0_a \quad \checkmark$

Compatibility of scalar $\overset{S.m}{\text{multiplication}}$ with field $\overset{f.m}{\text{multiplication}}$:

$$\alpha(\beta v_a) = \alpha(\beta v)_a = (\alpha(\beta v))_a = ((\alpha\beta)v)_a = (\alpha\beta)v_a \quad \checkmark$$

Identity element of scalar $\overset{S.m}{\text{multiplication}}$: $1\, v_a = (1v)_a = v_a \quad \checkmark$

Distributivity of scalar $\overset{S.m}{\text{multiplication}}$ with respect to vector $\overset{v.a}{\text{addition}}$:

$$\alpha(u_a + v_a) = \alpha(u+v)_a = (\alpha(u+v))_a = (\alpha u + \alpha v)_a$$

$$= (\alpha u)_a + (\alpha v)_a = \alpha u_a + \alpha v_a \quad \checkmark$$

Distributivity of scalar $\overset{S.m}{\text{multiplication}}$ with respect to field $\overset{f.a}{\text{addition}}$:

$$(\alpha+\beta)v_a = ((\alpha+\beta)v)_a = (\alpha v + \beta v)_a = (\alpha v)_a + (\beta v)_a$$

$$= \alpha v_a + \beta v_a \quad \checkmark$$

ب)

$$D_v\big|_a (f+g) = \frac{d}{dt}\Big|_{t=0} (f+g)(a+tv) = \frac{d}{dt}\Big|_{t=0} f(a+tv) + \frac{d}{dt}\Big|_{t=0} g(a+tv)$$

$$= D_v\big|_a f + D_v\big|_a g$$

$$D_v\big|_a (\alpha f) = \frac{d}{dt}\Big|_{t=0} (\alpha f)(a+tv) = \alpha \frac{d}{dt}\Big|_{t=0} f(a+tv) = \alpha\, D_v\big|_a (f)$$

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$$D_v\big|_a (fg) = \frac{d}{dt}\Big|_{t=0} (fg)(a+tv) = \overbrace{f(a+tv)\Big|_{t=0}}^{f(a)} \frac{d}{dt}\Big|_{t=0} g(a+tv) + \overbrace{g(a+tv)\Big|_{t=0}}^{g(a)} \frac{d}{dt}\Big|_{t=0} f(a+tv)$$

$$= f(a)\, D_v\big|_a (g) + g(a)\, D_v\big|_a (f)$$

ت)

$$D_v\big|_a f = \frac{d}{dt}\Big|_{t=0} f(a+vt) = \left[\left(\overbrace{\frac{dx}{dt}}^{v^1}\frac{\partial f}{\partial x} + \overbrace{\frac{dy}{dt}}^{v^2}\frac{\partial f}{\partial y} + \overbrace{\frac{dz}{dt}}^{v^3}\frac{\partial f}{\partial z}\right)(a+tv)\right]_{t=0}$$

$$= v^1\frac{\partial f}{\partial x}(a) + v^2\frac{\partial f}{\partial y}(a) + v^3\frac{\partial f}{\partial z}(a) \qquad \begin{cases} x = v^1 t + a^1 \\ y = v^2 t + a^2 \\ z = v^3 t + a^3 \end{cases}$$

ث)

$$v_a = e_1\big|_a \Rightarrow v^1 = 1,\ v^2 = 0\quad v^3 = 0$$

$$\xRightarrow{\text{با توجه به قسمت (ت)}} D_v\big|_a f = \frac{\partial f}{\partial x}(a)$$

ج) زیرا در قسمتهای ب و پ نشان دادیم که مشتق جهتی هم خطی است و هم قاعده لایب نیتس را ارضا می کند که این همان تعریف مشتق است.

چ) برای اثبات فضای برداری بودن $T_a(\mathbb{R}^3)$ باید موارد زیر را بررسی کنیم:

Associativity of addition: $(X+(Y+Z))f = Xf+(Y+Z)f = Xf+(Yf+Zf) = (Xf+Yf)+Zf$
$= (X+Y)f + Zf = ((X+Y)+Z)f$ ✓

Commutativity of addition: $(X+Y)f = Xf+Yf = Yf+Xf = (Y+X)f$ ✓

Identity element of addition: $\exists\, 0 \in T_a(\mathbb{R}^3)$ st $0f := 0 \in \mathbb{R}$
$(0+X)f = 0f + Xf = 0 + Xf = Xf$ ✓

Inverse elements of addition: $\forall X \in T_a(\mathbb{R}^3)\quad \exists\,(-X) \in T_a(\mathbb{R}^3)$ st $(-X)f := -Xf$
$(X+(-X))f = Xf + (-X)f = Xf + (-Xf) = (\overbrace{X-X}^{0})f = 0f$ ✓

Compatibility of s.m with f.m: $(\alpha(\beta X))f = \alpha(\beta X)f = \alpha(\beta(Xf)) = (\alpha\beta)(Xf) = ((\alpha\beta)X)f$ ✓

Identity element of s.m: $(1X)f = 1(Xf) = Xf$ ✓

Distributivity of s.m with respect to v.a: $(\alpha(X+Y))f = \alpha(X+Y)f = \alpha(Xf+Yf) = \alpha Xf + \alpha Yf$
$= (\alpha X + \alpha Y)f$ ✓

Distributivity of s.m with respect to f.a: $((\alpha+\beta)X)f = (\alpha+\beta)(Xf) = \alpha(Xf) + \beta(Xf)$
$= (\alpha X + \beta X)f$ ✓

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$$f_1(x)=1 \qquad X(f_1) = X(f_1\cdot f_1) = \underbrace{f_1}_{1}(Xf_1) + \underbrace{f_1}_{1}(Xf_1) = 2(Xf_1)$$

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$$\Rightarrow (Xf_1) = 2(Xf_1) \Rightarrow \boxed{Xf_1 = 0}$$

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$$f(x)=c \qquad X(f) = X(c\cdot f_1) = c\underbrace{(Xf_1)}_{0} = 0$$

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$$X(f(x,y,z)) = X(\cancelto{0}{f(a)}) + X\left(\frac{\partial f}{\partial x}(a)\,(x-a^1)\right) + X\left(\frac{\partial f}{\partial y}(a)\,(y-a^2)\right) + X\left(\frac{\partial f}{\partial z}(a)\,(z-a^3)\right)$$

$$X(g_1(x^1-a^1)) + X(g_2(y-a^2)) + X(g_3(z-a^3))$$

$$= X\left(\cancelto{0}{\frac{\partial f}{\partial x}(a)}\right)(x-a^1) + \frac{\partial f}{\partial x}(a)\,X(x-a^1) + X\left(\cancelto{0}{\frac{\partial f}{\partial y}(a)}\right)(y-a^2) + \frac{\partial f}{\partial y}(a)\,X(y-a^2)$$

$$+ X\left(\cancelto{0}{\frac{\partial f}{\partial z}(a)}\right)(z-a^3) + \frac{\partial f}{\partial z}(a)\,X(z-a^3) + (Xg_1)(x-a^1)\Big|_{x=a^1} + g_1\Big|_{(x,y,z)=a} X(x-a^1)$$

$$+ (Xg_2)(y-a^2)\Big|_{y=a^2} + g_2\Big|_{(x,y,z)=a} X(y-a^2) + (Xg_3)(z-a^3)\Big|_{z=a^3} + g_3\Big|_{(x,y,z)=a} X(z-a^3)$$

$$= \frac{\partial f}{\partial x}(a)\,X(x) + \frac{\partial f}{\partial y}(a)\,X(y) + \frac{\partial f}{\partial z}(a)\,X(z)$$

$$= v^1\frac{\partial f}{\partial x}(a) + v^2\frac{\partial f}{\partial y}(a) + v^3\frac{\partial f}{\partial z}(a) = D_v\big|_a f \Rightarrow \boxed{X = D_v\big|_a}$$

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$$F(v_a+w_a)f = D_{v+w}\big|_a f = (v^1+w^1)\frac{\partial f}{\partial x}(a) + (v^2+w^2)\frac{\partial f}{\partial y}(a) + (v^3+w^3)\frac{\partial f}{\partial z}(a)$$

$$= \left(v^1\frac{\partial f}{\partial x}(a) + v^2\frac{\partial f}{\partial y}(a) + v^3\frac{\partial f}{\partial z}\right) + \left(w^1\frac{\partial f}{\partial x}(a) + w^2\frac{\partial f}{\partial y}(a) + w^3\frac{\partial f}{\partial z}(a)\right)$$

$$= D_v\big|_a f + D_w\big|_a f = (F(v_a)+F(w_a))f \quad ✓$$

$$F(cv_a)f = D_{cv}\big|_a f = (cv^1)\frac{\partial f}{\partial x}(a) + (cv^2)\frac{\partial f}{\partial y}(a) + (cv^3)\frac{\partial f}{\partial z}(a)$$

$$= c\,D_v\big|_a f = (c\,F(v_a))f \quad ✓$$

و) $F(v_a) = F(w_a) \Rightarrow F(v_a) - F(w_a) = 0 \Rightarrow \big(F(v_a) - F(w_a)\big) f = 0$

$$\Rightarrow \big(F(v_a - w_a)\big) f = 0 \Rightarrow D_{v-w}\big|_a f = 0 \Rightarrow (v^1 - w^1)\frac{\partial f}{\partial x} + (v^2 - w^2)\frac{\partial f}{\partial y} + (v^3 - w^3)\frac{\partial f}{\partial z} = 0$$

$$\Rightarrow \begin{cases} v^1 = w^1 \\ v^2 = w^2 \\ v^3 = w^3 \end{cases} \Rightarrow v = w \Rightarrow$$ F یک به یک است.

ز) $F:\mathbb{R}^3_a \longrightarrow T_a(\mathbb{R}^3)$ پوشا است اگر به ازای هر $X \in T_a(\mathbb{R}^3)$ یک بردار مانند $v_a \in \mathbb{R}^3_a$ وجود داشته باشد به طوری که $F(v_a) = X$. در قسمت (د) به ازای هر X یک بردار معرفی کنید که این ویژگی را دارد.

موفق باشید